Second-order Darboux displacements

Second-order Darboux displacements

B F Samsonov[1,3]**, M L Glasser**[2]**, J Negro**[1] **and L M Nieto**[1]

[1] Departamento de Física Teórica, Universidad de Valladolid, 47005 Valladolid, Spain
[2] Department of Physics and Center for Quantum Device Technology, Clarkson University, Potsdam, NY 13699-5820, USA

E-mail: samsonov@phys.tsu.ru, laryg@clarkson.edu, jnegro@fta.uva.es and luismi@metodos.fam.cie.uva.es

Abstract
The potentials for a one-dimensional Schrödinger equation that are displaced along the x-axis under second-order Darboux transformations, called 2-SUSY invariant, are characterized in terms of a differential–difference equation. The solutions of the Schrödinger equation with such potentials are given analytically for any value of the energy. The method is illustrated by a two-soliton potential. It is proved that a particular case of the periodic Lamé–Ince potential is 2-SUSY invariant. Both Bloch solutions of the corresponding Schrödinger equation are found for any value of the energy. A simple analytic expression for a family of two-gap potentials is derived.

PACS numbers: 03.65.Ge, 03.65.Fd, 03.65.Ca

1. Introduction

Originally, it was noted in the context of periodic potentials [1, 2] that transformed Hamiltonians under special Darboux transformations are displaced with respect to the initial potential by half a period. Later on, this effect was studied in more detail with respect to first-order Darboux transformations (also called 1-SUSY transformations) [3, 4]. It was shown that the range of possible displacements produced by a Darboux transformation is not restricted to half a period, but can take on values in a set of (in general) complex values called 'Darboux displacements' (or SUSY displacements). The potentials allowing such displacement are called translationally invariant under the Darboux transformation or, simply, 1-SUSY invariant. It was proved that a real-valued even potential function $V_0(x) = V_0(-x)$ is 1-SUSY invariant if and only if it is of the form $V_0(x) = 2\wp(x + \omega')$, where $\wp(x) = \wp(x, g_2, g_3)$ is the standard Weierstrass elliptic function with real and imaginary half-periods ω and ω' and invariants g_2, g_3 [5] (including its degenerate forms such as $2x^{-2}$, $-2\cosh^{-2} x$ and $2\sinh^{-2} x$). This result

---

[3] On leave from Physics Department of Tomsk State University, 634050 Tomsk, Russia.

really means that the family of 1-SUSY invariant potentials is rather sparse. Moreover, it was noted in [3] that a simple displacement appears to be a 'frustrated case' of the Darboux method. Nevertheless, this property has led to mathematically nontrivial results and to unexpected link between the theory of elliptic functions and supersymmetric quantum mechanics [3, 4]. From a physical point of view a very remarkable property of 1-SUSY invariant potentials is that the corresponding Bloch solutions can be found analytically for any value of the energy. If a linear combination of Bloch solutions is used as the transformation function for a simple SUSY transformation, it produces an exactly solvable potential with locally perturbed periodic structure. We believe these potentials could find applications in describing contact effects in crystals, or modelling crystals with embedded inclusions.

In this paper we continue the investigation of SUSY invariant potentials, but at the level of second-order Darboux transformations. The aim of this work is to study what happens if a potential is assumed to be 2-SUSY invariant, i.e., a second Darboux transformation results only in a displacement $V_0(x) \to V_2(x) = V_0(x + d)$. From the very beginning it is clear that this condition is weaker than 1-SUSY invariance, since any 1-SUSY invariant potential is obviously also 2-SUSY invariant. Actually, as a preliminary result shows [4], the family of 2-SUSY invariant potentials is richer than that of 1-SUSY invariant ones; at least, it includes a class of 2-soliton potentials which are not 1-SUSY invariant. Indeed, as we show below, it is even much richer, since a simple Darboux transformation over such a potential gives another 2-SUSY invariant potential which cannot be reduced to a displaced copy of the initial one. Another remarkable property of a 2-SUSY invariant potential is that it allows for an analytic representation of both linearly independent solutions of the corresponding Schrödinger equation at any value of the energy (not necessarily from the spectral set). Using this property we are also able to generate new exactly solvable potentials with locally distorted periodic structure, as illustrated below. Our results lead us to hypothesize that the general Lamé–Ince potential is invariant under an nth order Darboux transformation.

This paper is organized as follows. In the next section we review several properties of Darboux transformations and 1-SUSY invariant potentials. In section 3 we prove necessary and sufficient conditions for a potential $V_0(x)$ to be 2-SUSY invariant and point out some simple implications of this property. Section 4 has purely illustrative character. Here we apply the results of the previous section to a well-studied family of 2-soliton potentials. Section 5 is devoted to an analysis of the $n = 2$ case of the Lamé–Ince potential $V_0(x) = n(n+1)\wp(x+\omega')$. It should be mentioned that this remarkable potential has attracted much attention from mathematicians as well as physicists. Without going into further details we refer the interested reader to the excellent recent papers [6] where a vast literature is summarized. We believe that our general results applied to the $n = 2$ case establish new properties of this remarkable potential. In particular, we show that it is 2-SUSY invariant and give a simple analytic representation of its linearly independent Bloch solutions in terms of the Weierstrass functions. Further we give an explicit formula for one-parameter family of two-gap (i.e. having only two finite forbidden and allowed bands) potentials. General properties of such potentials have been studied by algebraic–geometrical methods (see e.g. [6, 7]). The most striking of our developments is that we use only elementary means and the well-known properties of Weierstrass functions presented in [5].

2. Preliminaries

Darboux transformations (also known as SUSY-QM transformations) have become an important tool for dealing with spectral problems associated with the Schrödinger equation, especially in one spatial dimension. Though the basic procedure is quite simple, the method

has attracted increasing attention from mathematicians and physicists for more than a century (see e.g. [8, 9]).

One begins with a Hamiltonian

$$h_0 = -\partial^2 + V_0(x) \qquad \partial \equiv d/dx \qquad x \in \mathbb{R} \tag{2.1}$$

(in appropriate units) whose eigenspace is two-dimensional for any eigenvalue $a \in \mathbb{C}$:

$$h_0 u(a, x) = au(a, x) \qquad h_0 \tilde{u}(a, x) = a\tilde{u}(a, x). \tag{2.2}$$

As is well known, imposition of boundary, summability or Bloch-type constraints selects 'physical' solutions $\psi(E, x)$ and leads to physical interpretation of the eigenvalue E as a spectral parameter:

$$h_0 \psi(E, x) = E\psi(E, x) \qquad E \in \mathbb{R}. \tag{2.3}$$

If $u_1 \equiv u(a_1, x)$ is a real and nodeless solution of equation (2.2), then the *1-SUSY partner* Hamiltonian

$$h_1 = -\partial^2 + V_1(x) \qquad V_1(x) = V_0(x) - 2(\ln u_1)'' \tag{2.4}$$

(the prime denotes the derivative with respect to x) is isospectral with h_0. (For non-periodic $V_0(x)$ the spectrum of h_1 may change in one point, but we ignore this for the moment.) Furthermore, an *intertwining* operator L obeying $Lh_0 = h_1 L$ has the form

$$L = -\partial + (\log u_1)'. \tag{2.5}$$

Thus, the eigenfunctions of h_1 for eigenvalue $a \neq a_1$ are $v(a, x) = Lu(a, x)$, $h_1 v(a, x) = av(a, x)$. This procedure can be repeated by starting with h_1 to produce a second isospectral Hamiltonian $h_2 = -\partial^2 + V_2(x)$, etc. It can be shown that in this way, for fixed u_1 and $u_2 \equiv u(a_2, x)$, the transformed potential is expressed in terms of the Wronskian $W(x) \equiv W(u_1, u_2) = u_1 u_2' - u_1' u_2$ as

$$V_2(x) = V_0(x) - 2[\log W(x)]''. \tag{2.6}$$

For $a \neq a_1, a_2$ the eigenfunctions of h_2 are $v(a, x) = W(u_1, u_2, u)W^{-1}(u_1, u_2)$, $h_2 v(a, x) = av(a, x)$ where $u \equiv u(a, x)$. It is easy to see that $W(u_1, u_2, \tilde{u}_{1,2}) \propto u_{2,1}$ so that for $a = a_1, a_2$, up to an inessential constant factor, this formula gives

$$v_{1,2} \equiv v_{1,2}(a_{1,2}, x) = u_{2,1} W^{-1}(u_1, u_2) \qquad h_2 v_{1,2} = a_{1,2} v_{1,2}. \tag{2.7}$$

We note that this is a particular case of a much more general result for a chain of N Darboux transformations [9]. Solutions of the initial Schrödinger equation $u_j \equiv u_j(a_j, x)$ are called *transformation functions* while their eigenvalues a_j are known as *factorization constants*. We refer the reader to [9] for more details.

A particularly interesting case is where V_0 is periodic:

$$V_0(x + T) = V_0(x). \tag{2.8}$$

We recall that inside the eigenfunction space of each eigenvalue for such a Hamiltonian we can build up Bloch functions, i.e., two linearly independent solutions $u^\pm$ of $h_0 u^\pm = au^\pm$ such that

$$u^\pm(a, x + T) = [\beta(a)]^{\pm 1} u^\pm(a, x) \qquad \beta(a) \in \mathbb{C}. \tag{2.9}$$

When for $a = E$ one has $|\beta(E)| = 1$, the E-values belong to a spectral band and the corresponding ('physical') solutions are bounded, while for $|\beta(a)| \neq 1$ they are unbounded ('non-physical') and the values of a lay in forbidden bands (see e.g. [10]). By using appropriate Bloch functions as transformation functions in the Darboux algorithm one can construct

periodic partner Hamiltonians [1–4, 11]. A linear combination of Bloch functions leads to a perturbed periodic structure which is asymptotically periodic.

Let, for instance, the transformation functions be Bloch functions $u_1^+(x)$ and $u_2^-(x)$ with corresponding factors $\beta_{1,2} = \beta(a_{1,2})$ as defined in (2.9). Then, according to (2.7), Bloch solutions of the transformed equation corresponding to eigenvalues a_1 and a_2 are respectively

$$v_1^- = \frac{u_2^-}{W(x)} \qquad v_2^+ = \frac{u_1^+}{W(x)} \qquad h_2 v_{1,2}^{\pm} = a_{1,2} v_{1,2}^{\pm} \tag{2.10}$$

where

$$v_1^-(x + T) = \beta_1^{-1} v_1^-(x) \qquad v_2^+(x + T) = \beta_2 v_2^+(x). \tag{2.11}$$

If under a Darboux transformation we have

$$V_1(x) = V_0(x + d) \tag{2.12}$$

then d is called a *Darboux displacement* and V_0 is said to be *1-SUSY invariant*. The first cases of Darboux displacements were found for periodic potentials [1, 2]. Later it was proved [3] that an even function $V_0(x)$ allows for a first-order Darboux displacement if and only if it satisfies the nonlinear differential–difference equation

$$V_0(x) + V_0(x + d) - \frac{1}{2} \left[\frac{V_0'(x) + V_0'(x + d)}{V_0(x) - V_0(x + d)} \right]^2 = \text{const} \tag{2.13}$$

which, up to a constant, is equivalent to the addition formula for the Weierstrass $\wp$ function. This result means that the family of even 1-SUSY invariant potentials is restricted to the $\wp$ function and its degenerate forms, such as one-soliton potentials. In the next section we prove necessary and sufficient conditions for a potential V_0 to admit a displacement under a second-order Darboux transformation. The potentials possessing this property will be called *2-SUSY invariant*.

3. Second-order Darboux displacements

Let, as before, $u_1^+(x)$ and $u_2^-(x)$ be two Bloch eigenfunctions of h_0 which are chosen as transformation functions for a 2-SUSY transformation. Although we concentrate on the case of periodic potentials $V_0(x + T) = V_0(x)$, our results have a more general character, as will be mentioned below. Now we assume that the potential V_2, obtained according to (2.6), is displaced from the original one, i.e.,

$$V_2(x) = V_0(x + d). \tag{3.1}$$

Thus, the differential equations corresponding to h_0 and h_2 have exactly the same solutions, but shifted by the displacement d. Hence, taking into account property (2.11) we see that

$$v_1^-(x - d) \propto u_1^-(x) \qquad v_2^+(x - d) \propto u_2^+(x). \tag{3.2}$$

From this and (2.10) it follows readily that

$$W(x) = c_1 \frac{u_2^-(x)}{u_1^-(x + d)} = c_2 \frac{u_1^+(x)}{u_2^+(x + d)} \tag{3.3}$$

where $c_{1,2}$ are constants. It is convenient to use the notation

$$\tilde{x} = x + d \qquad \Phi(x) = 1/W(x) \qquad p(x) = -2[\ln \Phi(x)]'. \tag{3.4}$$

Therefore, $p'(x) = V_0(x) - V_0(\tilde{x})$ and from (3.3) it follows that

$$u_1^-(\tilde{x}) = c\Phi(x) u_2^-(x) \tag{3.5}$$

where c is an inessential non-zero constant. After taking the second derivative of (3.5) and using (2.2), one obtains

$$[V_0(\tilde{x}) - a_1]\Phi(x)u_2^-(x) = [V_0(x) - a_2]\Phi(x)u_2^-(x) + u_2^-(x)\Phi''(x) + 2[u_2^-(x)]'\Phi'(x). \quad (3.6)$$

Now, from (3.4) one has

$$\Phi' = -\frac{p}{2}\Phi \qquad \Phi'' = -\frac{p'}{2}\Phi + \frac{p^2}{4}\Phi \tag{3.7}$$

and from (3.6) this yields

$$[\ln u_2^-(x)]' = \frac{p'}{2p} + \frac{p}{4} + \frac{a_1 - a_2}{p}. \tag{3.8}$$

Similarly

$$\left[\ln u_1^+(x)\right]' = \frac{p'}{2p} + \frac{p}{4} - \frac{a_1 - a_2}{p} \tag{3.9}$$

$$\left[\ln u_2^+(\tilde{x})\right]' = \frac{p'}{2p} - \frac{p}{4} - \frac{a_1 - a_2}{p} \tag{3.10}$$

$$[\ln u_1^-(\tilde{x})]' = \frac{p'}{2p} - \frac{p}{4} + \frac{a_1 - a_2}{p}. \tag{3.11}$$

Remark 1. Formulae (3.8)–(3.11) illustrate one of the most remarkable properties of a 2-SUSY invariant potential. The four Bloch functions corresponding to the eigenvalues a_1 and a_2 producing a displacement are expressible only in terms of the potential difference $p'(x) = V_0(x) - V_0(x + d)$ (a known function) and its primitive (also known from equation (3.19)).

Later on for a particular case of the Weierstrass potential we will integrate these equations to get analytic expressions for the solutions of the Schrödinger equation.

We turn next to finding necessary and sufficient conditions for $V_0(x)$ to admit a displacement under a second-order Darboux transformation (shortly second-order Darboux displacement).

Proposition 1. *The Hamiltonian h_0 allows for a second-order Darboux displacement if and only if the potential $V_0(x)$ satisfies the following differential–difference equation:*

$$T - \frac{1}{4}\left[\frac{T'}{M}\right]^2 = \text{constant} \tag{3.12}$$

where

$$T = (S'/D)^2 + 4S - 2D''/D \qquad M = D - (S'/D)' \tag{3.13}$$

and

$$S = V_0(x) + V_0(\tilde{x}) \qquad D = V_0(x) - V_0(\tilde{x}). \tag{3.14}$$

Note that this formula resembles (2.13) for the first-order displacement, but the functions T and M involved depend in a much more complex way on $V_0(x)$.

Proof. Since $u'' = [(u'/u)' + (u'/u)^2]u$, the Schrödinger equation for u_2^- is

$$\left[\frac{(u_2^-)'}{u_2}\right]' + \left[\frac{(u_2^-)'}{u_2}\right]^2 + a_2 - V_0(x) = 0 \tag{3.15}$$

which together with (3.8) results in

$$2pp'' - (p')^2 + \tfrac{1}{4}p^4 + 4(a_1 - a_2)^2 + 2p^2(a_1 + a_2 - S) = 0 \qquad (3.16)$$

where S is defined in (3.14). Now we take the derivative of (3.16) and, since $p' = V_0(x) - V_0(x + d)$ is a known function, we can treat the result as a quadratic equation for p, whose solution is

$$p = \frac{S'}{p'} \pm \sqrt{\left(\frac{S'}{p'}\right)^2 - 2\frac{p'''}{p'} + 4S - 4(a_1 + a_2)}. \qquad (3.17)$$

Then taking the derivative of (3.17), and rearranging, we obtain the nonlinear differential–difference equation for $V_0(x)$

$$4(a_1 + a_2) = T - \frac{1}{4}\left[\frac{T'}{D - (S'/D)'}\right]^2 \qquad (3.18)$$

where T is defined by (3.13). Keeping in mind that the eigenvalues a_1 and a_2 are independent of x and taking into consideration the definition of M (second formula in (3.13)), we see that the necessary condition is fulfilled. We shall show now that this condition is also sufficient.

First, we note that by using (3.18) to eliminate $a_1 + a_2$ in (3.17) we find

$$p = \frac{S'}{D} + \frac{1}{2}\frac{T'}{D - (S'/D)'}. \qquad (3.19)$$

Here we have to choose only plus sign in (3.17) since it is easy to see that the minus contradicts the condition $p' = V_0(x) - V_0(x + d)$. If the right-hand side of (3.18) is constant, then from (3.19) we obtain p which, by retracing our steps, satisfies (3.16) which is equivalent to the Schrödinger equation (3.15) for u_2^-. Therefore, from (3.8) and (3.9), we have the logarithmic derivatives of u_2^- and u_1^+. These being known, we are able to calculate the second logarithmic derivative of their Wronskian

$$[\ln W]'' = (a_1 - a_2)\frac{\left[\ln\left(u_1^+ u_2^-\right)\right]'}{\left[\ln\left(u_1^+/u_2^-\right)\right]'} - \left(\frac{a_1 - a_2}{\left[\ln\left(u_1^+/u_2^-\right)\right]'}\right)^2 = \frac{1}{2}D \qquad (3.20)$$

which by (2.6) means that h_0 possesses a second-order Darboux displacement. $\qquad \square$

Corollary 1. *Any 2-SUSY invariant potential allowing a set of displacements d generates a one-parameter family of 2-SUSY invariant potentials (some of them may have singular points) obtained by a 1-SUSY transformation with one of the four functions $u_1^\pm, u_2^\pm$ as transformation function.*

Proof. Let $V_0(x)$ be 2-SUSY invariant. This means that there exists a second-order Darboux transformation with transformation functions u_1^- and u_2^+ which results only in a displacement of the transformed potential $V_2(x) = V_0(x+d)$. Any second-order transformation may be factored into two successive first-order transformations. One can realize the first transformation with the transformation function u_1^- to get an intermediate potential $V_1(x)$ which cannot be obtained by a displacement of $V_0(x)$ if the later is not 1-SUSY invariant and hence $V_1(x)$ is essentially different of $V_0(x)$. We shall show that $V_1(x)$ allows for the same second-order Darboux displacement. In order to show this property take the logarithms of the first equality in (3.3)

$$2\log W(x) + 2\log u_1^-(x + d) = 2\log c_1 + 2\log u_2^-(x).$$

After taking two derivatives and using (2.6) and (3.1) we obtain

$$V_0(x) - 2[\log u_2^-(x)]'' = V_0(x + d) - 2[\log u_1^-(x + d)]''. \qquad (3.21)$$

The left-hand side of this equality means that we realize a 1-SUSY transformation of the potential V_0 with the transformation function $u_2^-(x)$ and from the right-hand side we learn that a similar transformation is realized of the shifted version of the same potential with the transformation function $u_1^-(x + d)$. Since the Darboux transformations are invertible, we see that these transformed potentials may always be related to each other with the help of a 2-SUSY transformation for which V_0 is an intermediate potential. Hence, equality (3.21) means that after such a transformation we get only a displacement by the value d. If the function $u_2^-(x)$ has nodes, the 1-SUSY partner of $V_0(x)$ will have singularities (poles). To displace $V_0(x)$ by another value we have to choose a second pair of transformation functions u_1^+ and u_2^- that give us another member of the family of 2-SUSY invariant potentials. $\square$

Remark 2. No member of the family obtained above is a displaced copy of the initial potential if it is not 1-SUSY invariant. Moreover, few of the potentials are displaced copies of other members of the same family, which means that the whole set of 2-SUSY invariant potentials is quite rich.

Indeed, if the initial potential is not 1-SUSY invariant it is not possible to displace it by a 1-SUSY transformation. Next, any two different members of this family are related by a 2-SUSY transformation, where V_0 is an intermediate potential. The factorization parameters of such a transformation are independent of each other, which means that once a representative of the family is fixed, $\tilde{V}$, the whole family is characterized by two independent (factorization) parameters. The potential $\tilde{V}$, being 2-SUSY invariant, also admits 2-SUSY transformations resulting only in a displacement, but these transformations are characterized by one parameter only. This follows from equations (3.16) and (3.18) which may be considered as defining one of the factorization constants (say a_2) as an implicit function of the other (a_1). This property will also be illustrated in the next section and used in section 5 to produce new exactly solvable periodic potentials.

Remark 3. The results of this section are valid not only for a periodic potential but for any potential satisfying the conditions of proposition 1.

This follows from the fact that the only place where we used the Bloch property of solutions of the Schrödinger equation (that is the periodicity of the potential $V_0(x)$) is in formula (3.2). Thus, $u_1^{\pm}$ and $u_2^{\pm}$ can be not only Bloch solutions, but also two pairs of linearly independent solutions of the Schrödinger equation with a non-periodic potential.

A simple sufficient condition for (3.12) is that $T = (S'/D)^2 + 4S - 2D''/D$ be constant. For example, consider the 1-SUSY invariant potential $V_0(x) = 2\wp(x - \omega')$. From the differential equation for $\wp$, we obtain $V_0'' = 3V_0^2 - g_2$ and therefore $D'' = 3V_0''(x) - 3V_0''(x + d) = 3DS$. From (2.13), the 1-SUSY invariance requires $(S'/D)^2 = 2S + \text{const}$. Therefore, in this case, $T = (S'/D)^2 - 2S = \text{const}$ and, consequently, V_0 is also 2-SUSY invariant, which of course was evident from the outset.

Note that for even potentials, $V_0(-x) = V_0(x)$, the expression in square brackets in (3.12) is not defined for $x = -d/2$. Using Mathematica [12] we have found that in this case

$$\lim_{x \to d/2} \frac{1}{4} \left[\frac{T'}{D - (S'/D)'} \right]^2 = \frac{12(V_0')^3 V_0'' + (V_0'')^2 V_0''' - V_0'[(V_0''')^2 + V_0'' V_0^{(IV)}] + (V_0')^2 V_0^{(V)}}{V_0'[6(V_0')^3 + V_0'' V_0''' - V_0' V_0^{(IV)}]}$$

where all the functions in the right-hand side are evaluated at $x = d/2$.

4. Two-soliton potentials

In this section we illustrate our previous developments for a well-studied two-soliton potential. In its most general form this potential is determined by four parameters:

$$V_0(x) = \frac{2\left(\alpha_1^2 - \alpha_2^2\right)\left(\alpha_1^2 \operatorname{sech}^2(\alpha_1 x + \beta_1) + \alpha_2^2 \operatorname{csch}^2(\alpha_2 x + \beta_2)\right)}{[\alpha_1 \tanh(\alpha_1 x + \beta_1) - \alpha_2 \coth(\alpha_2 x + \beta_2)]^2}. \qquad (4.1)$$

The parameters $\alpha_2 > \alpha_1 > 0$ define the positions of the discrete levels $E_{0,1} = -\alpha_{2,1}^2$, while β_1 and β_2 characterize isospectral deformations. It has been shown by direct calculation [4] that the symmetrical case, $\beta_1 = \beta_2 = 0$, is 2-SUSY invariant. We note that exactly the same calculations can be carried out for the general case, indicating that for any fixed values of the parameters $\alpha_{1,2}$ and $\beta_{1,2}$ the potential (4.1) is 2-SUSY invariant. This illustrates remark 3. With evident modifications of the results of [4] we give the solutions u_1^+ and u_2^- producing the second-order Darboux displacement d of V_0:

$$u_{1,2}^{\pm} = W(u_{10}, u_{20}, u_{30,40}) W^{-1}(u_{10}, u_{20})$$

where $u_{10} = \cosh(\alpha_1 x + \beta_1)$, $u_{20} = \sinh(\alpha_2 x + \beta_2)$ are solutions of the free particle Schrödinger equation generating the potential (4.1) from the zero potential, and $u_{30,40} = \exp(\alpha_{3,4} x)$, $\alpha_{3,4} > 0$. When parameters α_3 and α_4 vary in a certain domain, independently of each other, these functions produce an isospectral deformation of V_0, resulting in the same expression with the different values of β_1 and β_2. To select from these transformations only ones leading to the shift $V_0(x) \to V_2(x) = V_0(x + d)$ one has to impose an additional restriction on α_3 and α_4. The analysis in [4] shows that to get real displacements one has to change the factorization constants α_3 and α_4 so that they fall between the existing discrete levels of V_0: $\alpha_1 < \alpha_3 < \alpha_4 < \alpha_2$.

Let us now characterize the one-parameter family mentioned in corollary 1. For this purpose we have to realize a 1-SUSY transformation of the potential V_0 by using one of $u_{1,2}^{\pm}$ as the transformation function. For simplicity let us fix $\beta_1 = \beta_2 = 0$ and choose the function u_1^+ to produce the potential V_1. According to properties of Darboux transformations (see e.g. [9]) this is equivalent to a third-order transformation of the zero potential with transformation functions u_{10}, u_{20} and u_{30} or simply to a chain of transformations where, for instance, the first transformation is realized with the function u_{30}. For such a transformation the zero potential is not affected, but the functions u_{10} and u_{20} receive changes and, up to a constant factor, become $u_{10}(x) \to v_{10}(x) = \cosh(\alpha_1 x + \gamma_1)$, $u_{20}(x) \to v_{20}(x) = \sinh(\alpha_2 x + \gamma_2)$, where

$$\gamma_1 = \operatorname{arctanh}(\alpha_1/\alpha_3) \qquad \gamma_2 = \operatorname{arctanh}(\alpha_2/\alpha_3). \qquad (4.2)$$

After the second-order transformation with the functions v_{10} and v_{20} we get from the zero potential the same two-soliton potential (4.1) where β are replaced by γ. This is just the one-parameter family of corollary 1 where α_3 is the parameter. From definition (4.2) we see that the parameters γ_1 and γ_2 are not independent of each other: $\tanh \gamma_1 / \tanh \gamma_2 = \alpha_1/\alpha_2$. They vary so that the potential V_1 cannot be reduced to a displaced copy of the potential (4.1) at $\beta_1 = \beta_2 = 0$, since the later condition requires $\alpha_1/\alpha_2 = \gamma_1/\gamma_2$, which leads to $\tanh \gamma_1 / \tanh \gamma_2 = \gamma_1/\gamma_2$. The later is possible only for $\gamma_1 = \gamma_2$ which contradicts to $\alpha_1 \neq \alpha_2$. Just from this, the potential V_1, being of course a two-soliton potential, is essentially different from V_0, as pointed out in remark 2.

We will illustrate other properties derived in the previous section for the simplest case of this potential. We choose $\alpha_1 = 1$ and $\alpha_2 = 2$ to produce the well-known Pöschl–Teller

potential well

$$V_0(x) = -6 \operatorname{sech}^2 x \tag{4.3}$$

for which $E_0 = -4$ and $E_1 = -1$.

From (3.13) we find the value of T,

$$T = 4[-1 + \operatorname{csch}^2 d - 4 \operatorname{sech}^2 x - 4 \operatorname{sech}^2(d + x)] \tag{4.4}$$

which after being substituted into equation (3.18) gives the x-independent quantity

$$a_1 + a_2 = -5 - 3 \operatorname{csch}^2 d. \tag{4.5}$$

This means that the necessary and sufficient conditions for V_0 (4.3) to be 2-SUSY invariant are fulfilled. Now from (3.19) we obtain $p(x) = -6[\tanh x + \operatorname{csch} d \cosh x \operatorname{sech}(x + d)]$. From equation (3.16) we deduce the difference of the factorization constants

$$a_1 - a_2 = \pm\bigl(3 \coth d \sqrt{1 - 3 \operatorname{csch}^2 d}\bigr). \tag{4.6}$$

From here and (4.5) one gets one factorization constant as a function of the other: $a_1 = \frac{1}{2}\bigl(-6 - a_2 \pm \sqrt{-12a_2 - 3a_2^2}\bigr)$. It is clear that this equation and (4.5) define the displacement d as a function of a_2:

$$d = \operatorname{arccsch} \frac{\sqrt{-4 - a_2 + \sqrt{3}\sqrt{-a_2(a_2 + 4)}}}{\sqrt{6}}. \tag{4.7}$$

It follows from here that real displacements are possible only when both factorization constants lie between discrete levels, $-4 < a_{1,2} < -1$, and in the above formula for a_1 the lower sign is chosen. This agrees completely with the previous result [4]. We see also that when $a_1 \to -3$, then $a_2 \to -3$, and we get the minimal value of the displacement $d_{\min} = \operatorname{arccsh}(1/\sqrt{3})$. The other limit is $a_1 \to -4$, which gives $a_2 \to -1$, so that the displacement increases indefinitely, $d \to \infty$. In this respect we remark that for one-soliton potentials the (first-order Darboux) displacement range is always $(0, \infty)$.

We are now able to find the right-hand side of equation (3.8). It is expressed in terms of elementary functions only and its primitive gives us a solution of the Schrödinger equation with the potential V_0:

$$u_2^-(x) = \cosh(x + d)\, e^{-\frac{1}{2}(3\coth d + \sqrt{1 - 3\operatorname{csch}^2 d})}[3 - \cosh 2d + \cosh 2x + \cosh (2x + 2d)]^{1/2}$$

$$\times \operatorname{sech}^2 x \left[\frac{\sqrt{2}(\coth d + 2 \tanh x) \sinh d + \sqrt{\cosh 2d - 7}}{-\sqrt{2}(\coth d + 2 \tanh x) \sinh d + \sqrt{\cosh 2d - 7}} \right]^{1/2}. \tag{4.8}$$

The derivative of this expression yields the potential difference as a function of the parameter d:

$$\Delta V = [(4\cosh x \operatorname{csch} d \operatorname{sech}(x + d) + \tanh x)(\operatorname{sech}^2(x + d) \tanh(x + d) + 2\operatorname{sech}^2 x \tanh x)$$

$$+ (\operatorname{sech}^2(x + d) - \operatorname{sech}^2 x)(3\operatorname{csch}^2 d + \coth d \sqrt{1 - 3\operatorname{csch}^2 d} - \operatorname{sech}^2 x$$

$$- 2\operatorname{sech}^2(x + d) + 3 \tanh^2 x)][\tanh x + \operatorname{csch} d \cosh x \operatorname{sech}(x + d)]^{-2}. \tag{4.9}$$

Of course, this intermediate quantity can be eliminated from the final expression to obtain a somewhat simpler formula for the family of two-soliton potentials:

$$V_1(x) = 12[4 + 9a_2 - a_2^2 - 6 \operatorname{sech}^2 x + 12\sqrt{a_2} \tanh x - (a_2 - 1)((a_2 + 4) \cosh 2x$$

$$+ 4\sqrt{a_2} \sinh 2x)][a_2 - 4 + (a_2 + 2) \cosh 2x + 3a_2 \sinh 2x]^{-2}. \tag{4.10}$$
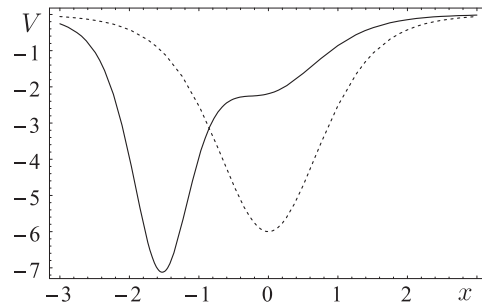
Figure 1. Two-soliton Pöschl–Teller potential (dashed line) and its SUSY partner at $a_2 = -4.1$.

For the potential difference to be regular one has to choose the factorization constant a_2 to be less than the ground state level of V_0, $a_2 < -4$, which gives complex values for the parameter d. Despite this, the potential difference is real and the solution (4.8) is also real (and nodeless) if it is set real at any point, i.e. $u_2^-(0) = 1$. As an example we have plotted one of the potentials $V_1 = V_0 + \Delta V$ with ΔV as in (4.9) in figure 1 (solid line) along with V_0 (dashed line).

5. The Lamé–Ince potential

A much more interesting example of second-order Darboux displacements is given by the following result.

Proposition 2. *The periodic two-gap potential*

$$V_0(x) = 6\wp(x + \omega') \tag{5.1}$$

where ω' and ω are the imaginary and real half-periods of the Weierstrass function $\wp$, with g_2 and g_3 as invariants, allows for second-order Darboux displacements. The eigenvalues $a_{1,2}$ of the transformation functions producing the displacement d are given by

$$a_{1,2} = -\tfrac{3}{2}\big[\wp(d) \mp \sqrt{g_2 - 3\wp^2(d)}\big]. \tag{5.2}$$

Proof. The proof consists simply in calculating the left-hand side of equation (3.12). We obtain first some useful relations between our variables which are direct implications of the well-known properties of the Weierstrass functions (see e.g. [5]). From the differential equation for $\wp$ one gets $D'' = SD$, so $T = (S'/D)^2 + 2S$. The addition formula for $\wp$ is

$$\wp(u + v) + \wp(u) + \wp(v) = \frac{1}{4}\left[\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)}\right]^2. \tag{5.3}$$

By setting $u = x + \omega' + d$, $v = -x - \omega'$ and noting that $\wp$ is even, this translates into $(S'/D)^2 = \tfrac{2}{3}S + 4\wp(d)$ and $T = \tfrac{8}{3}S + 4\wp(d)$. By taking the square root of the former of these relations and differentiating, we get

$$\frac{1}{D}\left(\frac{S'}{D}\right)' = \frac{1}{3} \tag{5.4}$$

and $T' = 8S'/3$, $D - (S'/D)' = 2D/3$. Inserting these relations into (3.17) yields

$$p = 3(S'/D). \tag{5.5}$$

Now we are able to calculate the left-hand side of (3.12) which gives

$$a_1 + a_2 = -3\wp(d). \tag{5.6}$$

This proves the first part of the statement. Now we shall find the eigenvalues of the transformation functions. For this purpose we shall use equation (3.16). Since $[\wp'(x)]^2 = 4\wp^3(x) - g_2\wp(x)$, we have the algebraic identity

$$D^2 - 6\frac{S'D'}{D} + 3S^2 = 36g_2 \tag{5.7}$$

and since $p = 3S'/D$, inserting it and (5.6) in (3.16) and taking into account (5.7) we obtain

$$(a_1 - a_2)^2 = 9[g_2 - 3\wp^2(d)]. \tag{5.8}$$

This equality together with (5.6) proves the statement. One can check that formulae (5.6) and (5.8), for the sum and difference of factorization constants, reduce to those for the specific case of the two-soliton potential (4.5) and (4.6), respectively, when we take the half-periods $\omega' = i\pi/2$ and $\omega = \infty$. $\qquad\square$

Some straightforward consequences of proposition 2 are:

Corollary 2. *Three of the five band edges for the potential (5.1),*

$$E_1 = 3e_3 \qquad E_{1'} = 3e_2 \qquad E_2 = 3e_1 \tag{5.9}$$

correspond to $d = \omega$, $d = \omega + \omega'$ and $d = \omega'$, where

$$e_1 = \wp(\omega) \qquad e_2 = \wp(\omega + \omega') \qquad e_3 = \wp(\omega'). \tag{5.10}$$

The lowest $E_0 = -\sqrt{3g_2}$ and the highest $E_{2'} = \sqrt{3g_2}$ band edges are stationary points for the factorization constants a_1 and a_2 as functions of the displacement d.

Corollary 3. *The pair of (real) factorization constants which give rise to a real displacement by means of a second-order Darboux transformation belong to the first finite forbidden band $[E_1, E_{1'}]$. The real displacements so produced are in the interval $(d_{\min}, d_{\max})$, where*

$$d_{\min} = \wp^{-1}\left(\sqrt{g_2/3}\right) \qquad d_{\max} = \omega. \tag{5.11}$$

$\wp^{-1}(t)$ is the function inverse to $t = \wp(z)$ for which an appropriate sheet has to be chosen; $d_{\max} = \omega$ is realized for $a_2 = E_1$ and $a_1 = E_{1'}$, while $d_{\min}$ is realized for $a_1 = a_2 = -\sqrt{3g_2}/2$.

It is interesting to note that $-e_k$ are the positions of the band edges for the one-gap potential $V_0 = -2\wp(x + \omega')$ and that in this case the range of the (first-order) Darboux displacements is $(0, \infty)$.

Proof. The first part of corollary 2 follows immediately from proposition 2 and known properties of the quantities e_k given in [5] (vol 2), when one uses $d = \omega$, $d = \omega + \omega'$ and $d = \omega'$. The second one is a direct consequence of (5.2) and conditions $a'_{1,2}(d) = 0$. Corollary 3 follows from the restriction on d to be such that $g_2 - 3\wp^2(d)$ be positive, since according to proposition 2 it is inside of the square root. $\qquad\square$

It is not difficult to see that for $d = \omega$, the factorization constants are $a_2 = E_1$ and $a_1 = E_{1'}$. This is the only possibility for the transformed potential to be displaced by half a real period of the Weierstrass $\wp$ function. The possibility for the potential (5.1) to be displaced by a half-period was previously noted in [1, 2]. Here we indicate a possibility of displacing the argument of the potential (5.1) by a complex value and, in particular, by half an imaginary period. We remark that the values for the band edges are given in [2] for the Jacobi form of the Schrödinger equation, which we relate to the Weierstrass form in appendix A.

We are now able to integrate equations (3.8)–(3.11) and find analytic expressions for the Bloch functions, as it is established in the following proposition.

Proposition 3. *The functions*

$$u_2^-(x) = \pm\sqrt{p}\left(\frac{\sigma(x+\omega'+d)}{\sigma(x+\omega')}\right)^{3/2}\left(\frac{\sigma(x-x_1)}{\sigma(x-x_2)}\right)^{1/2} e^{(b-\frac{3}{2}\zeta(d))x} \quad (5.12)$$

$$u_2^+(\tilde{x}) = \frac{p(x)}{u_2^-(x)} \quad (5.13)$$

are the Bloch eigenfunctions for the Hamiltonian (5.1) with the eigenvalue a_2, and

$$u_1^-(\tilde{x}) = u_2^-(x)\left(\frac{\sigma(x+\omega')}{\sigma(x+\omega'+d)}\right)^3 e^{3\zeta(d)x} \quad (5.14)$$

$$u_1^+(x) = \frac{p(x)}{u_1^-(\tilde{x})} \quad (5.15)$$

are the Bloch eigenfunctions with the eigenvalue a_1. Here σ and ζ are standard Weierstrass functions,

$$x_1 = \wp^{-1}\left(-\tfrac{1}{2}\wp_0 + \tfrac{1}{2}\sqrt{g_2 - 3\wp_0^2}\right) - \omega'$$
$$x_2 = \wp^{-1}\left(-\tfrac{1}{2}\wp_0 - \tfrac{1}{2}\sqrt{g_2 - 3\wp_0^2}\right) - \omega' \quad (5.16)$$

$\wp_0 = \wp(d)$, $\wp^{-1}$ *is the function inverse to $\wp$ and*

$$b = \tfrac{1}{2}\zeta(\omega' - x_2) - \tfrac{1}{2}\zeta(\omega' - x_1). \quad (5.17)$$

The proof can be found in appendix B.

Remark 4. The solutions given in (5.12) and (5.13) are valid for any eigenvalue a_2. By using (5.2) one can find a_1 as a function of a_2:

$$a_1 = \frac{-a_2 \pm \sqrt{9g_2 - 3a_2^2}}{2}. \quad (5.18)$$

Then with the aid of (5.6) d is expressed in terms of a_2 also. Hence, the Bloch functions (5.12) and (5.13) are determined only by the eigenvalue a_2 which can take any value and, thus, these functions generate the one-parameter family of 2-SUSY invariant potentials mentioned in corollary 1. With appropriate modifications the same is true for the functions (5.14) and (5.15).
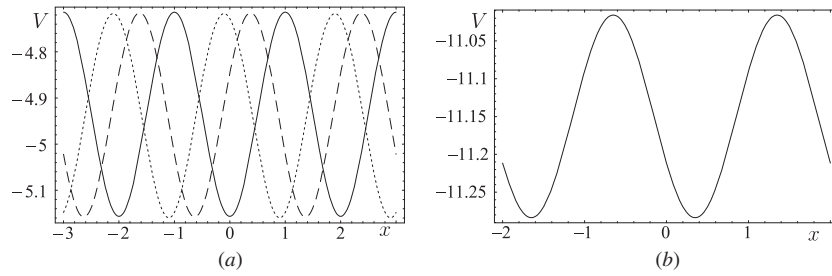
To illustrate this remark we prove:

Proposition 4. *The potentials*

$$V_1(x) = V_0(x) + \Delta V(x) \qquad V_0(x) = 6\wp(x+\omega') \quad (5.19)$$

where

$$\Delta V(x) = -\frac{[\wp(x+\omega'+d) + \wp(x+\omega')]^2 + 2\wp(x+\omega'+d)\wp(x+\omega') - g_2/2}{\wp(x+\omega'+d) + \wp(x+\omega') + \wp(d)}$$
$$+ 3\wp(x+\omega'+d) - 3\wp(x+\omega') - \wp(x-x_1) + \wp(x-x_2) \quad (5.20)$$

$d = \wp^{-1}\left(-\frac{a_1+a_2}{3}\right)$, x_1, x_2 *are defined as in (5.16), and a_1 in (5.18), form a one-parameter family of real-valued two-gap potentials isospectral with V_0. The role of the parameter is played by $a_2 \leqslant E_0$, where $E_0 = -\sqrt{3g_2}$ is the lowest band edge for V_0.*

Figure 2. (*a*) Lamé–Ince potential (solid line) at $\omega = 1$ and $\omega' = 2i$ and its SUSY partners. Dotted line corresponds to $a_2 = -5$ and dashed line to $a_2 = -6$; (*b*) illustrates that the left-hand side of equation (2.13) is not constant for $d = 1.3$.
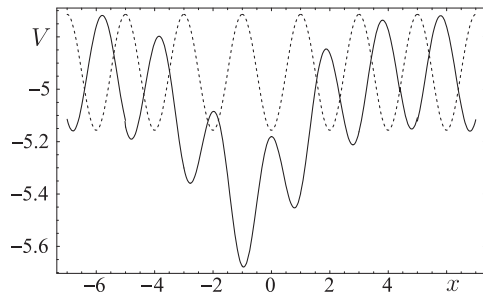
Figure 3. Potential corresponding to the transformation function $u = u_2^- + 0.5u_2^+$ at $\omega = 1$ and $\omega' = 2i$ and $a_2 = -5.2$ (solid line). Dashed line shows the initial Lamé–Ince potential.

Proof. The proof consists simply in calculating the right-hand side of equation (3.8) and taking its derivative which will give us the potential difference (see (2.4)). Using expression (B.2) below for p in terms of the Weierstrass ζ function and the addition formula (5.3) we first get

$$\frac{1}{2}(\log p)' = \frac{(\wp - \tilde{\wp})^2}{\wp' + \tilde{\wp}'} \tag{5.21}$$

where $\wp \equiv \wp(x + \omega')$ and $\tilde{\wp} \equiv \wp(x + \omega' + d)$. To derive the final expression (5.20) we have used the formula (B.8) and the following properties of the Weierstrass functions: $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$, $\wp'' = 6\wp^2 - g_2/2$ and $\zeta'(x) = -\wp(x)$. The fact that these potentials have two-gaps follows from the fact that Darboux transformations preserve band structure (see e.g. [11]). $\qquad\square$

It is interesting to observe that different representatives of this family look like displaced copies of the initial potential (see figure 2(*a*)) though V_0 is not 1-SUSY invariant which is clearly seen from figure 2(*b*), where the left-hand side of equation (2.13) is plotted.

Figure 3 shows a non-periodic potential obtained when a linear combination of the functions (5.12) and (5.13) is chosen as the transformation function. We stress that it possesses an energy level at -5.2 and the solution of the corresponding Schrödinger equation can easily be obtained by applying the first-order Darboux transformation operator (2.5) to the functions (5.12)–(5.15). We would also like to mention that the transformed potential tends to a displaced version of the initial potential at large values of $|x|$. This effect, first noted in [3], illustrates a nonlocal deformation produced by a Darboux transformation.

Finally, we would like to point out that the function (5.12) acquires a constant factor when its argument increases by the period 2ω. This can be seen from properties of the Weierstrass σ functions [5] that lead to

$$u_2^-(x + 2\omega) = u_2^-(x) \exp[\zeta(\omega)(3d + x_2 - x_1) - \omega(3\zeta(d) - 2b)].$$

From here it follows, in particular, that for $a_2 = 3e_3$, $a_1 = 3e_2$, one has $d = \omega$, $x_1 = \omega$ and $x_2 = -2\omega'$ which results in the condition $u_2^-(x + 2\omega) = u_2^-(x) \exp[2\zeta(\omega')\omega - 2\omega'\zeta(\omega)] = -u_2^-(x)$, where we have used the Legendre relation [5]. This result agrees with the fact that this function corresponds to the second band edge.

In conclusion, we have characterized and investigated the interesting class of periodic potentials which are merely translated by a second-order Darboux transformation. We have presented several examples, concentrating on the fascinating family of two-gap Lamé–Ince potentials. The nonlinear differential equation which must be satisfied by all such potentials, which may be related to the class of Painlevé equations, deserves further investigation.

Acknowledgments

Appendix A

Here, to fix our notation, we relate the Weierstrass form of the Schrödinger equation to its Lamé form (see also [5], vol 3). According to [5] one has

$$\text{sn}^2(z, k) = \frac{e_1 - e_3}{\wp(x) - e_3} \qquad \wp(x + \omega') = e_3 + \frac{(e_3 - e_2)(e_3 - e_1)}{\wp(x) - e_3} \qquad (A.1)$$

or

$$\wp(x + \omega') = e_3 + (e_2 - e_3)\text{sn}^2(z, k) \qquad (A.2)$$

where

$$k^2 \equiv m = \frac{e_2 - e_3}{e_1 - e_3} \qquad z = (e_1 - e_3)^{1/2}x. \qquad (A.3)$$

Now, after replacing the variable x by z in the Schrödinger equation with the potential (5.1), one gets

$$\left[-\frac{d^2}{dz^2} + (6k^2\text{sn}^2(z, k) - \tilde{E})\right]\psi = 0 \qquad (A.4)$$

where

$$\tilde{E} = \frac{E - 6e_3}{e_1 - e_3} = \frac{E}{e_1 - e_3} + 2m + 2. \qquad (A.5)$$

Using this relation and corollary 2 one recovers the known band edges for the Lamé equation (see e.g. [2]),

$$\tilde{E}_0 = \frac{E_0 - 6e_3}{e_1 - e_3} = 2m + 2 - 2\delta$$

$$\tilde{E}_1 = m + 1 \qquad \tilde{E}_{1'} = 4m + 1 \qquad (A.6)$$

$$\tilde{E}_2 = m + 4 \qquad \tilde{E}_{2'} = 2m + 2 + 2\delta.$$

Appendix B

In this appendix we prove proposition 3. In our notation the addition theorem for the Weierstrass ζ-function (see [5])

$$\frac{1}{2}\frac{\wp'(x+d)+\wp'(x)}{\wp(x+d)-\wp(x)} = \zeta(x) - \zeta(x+d) + \zeta(d) \tag{B.1}$$

becomes

$$\frac{1}{2}\frac{S'}{D} = \zeta(x+\omega'+d) - \zeta(x+\omega') - \zeta(d).$$

Hence

$$p = 3\frac{S'}{D} = 6[\zeta(x+\omega'+d) - \zeta(x+\omega') - \zeta(d)]. \tag{B.2}$$

Recalling that

$$p^2 = 9\left(\frac{S'}{D}\right)^2 = 6S + 36\wp(d) \tag{B.3}$$

we find

$$S = 6[\wp(x+\omega') + \wp(x+\omega'+d)]. \tag{B.4}$$

Let us find now the expression for $1/p$.

The function S has second-order poles at the same points as $\wp(x+\omega')$ and $\wp(x+\omega'+d)$, that is at $x = -\omega'$ and $x = -\omega' - d$. This means that S is a second-order elliptic function and has two second-order zeros. Here and below, for simplicity, we suppose that d falls in the interval $(d_{\min}, d_{\max})$ indicated in corollary 3.

Let us denote

$$f(x) := \wp(x+\omega') + \wp(x+\omega'+d) + \wp(d). \tag{B.5}$$

Then $p^2 = 36f \geqslant 0$ for real values of $\wp$. Note that for $d = \omega$, $f(0) = f(\omega) = e_1 + e_2 + e_3 = 0$. Hence, $x = 0$ and $x = \omega$ are the points of local minima for $f(x)$. Therefore $f'(0) = f'(\omega) = 0$, i.e. the zeros are of second order. Next, it is easy to see that $f'(\omega/2) = f'(3\omega/2) = 0$ and the points $x = \omega/2$ and $x = 3\omega/2$ are the points of local maxima for real valued $f(x)$. We find now the positions of these zeros.

Let x_0 be a minimum (or a zero-point) for $f(x)$, that is $f(x_0) = 0$ and $f'(x_0) = 0$. Let us abbreviate the notation by putting

$$\wp := \wp(x_0 + \omega') \qquad \tilde{\wp} := \wp(x_0 + \omega' + d) \qquad \wp_0 := \wp(d).$$

Then $\wp + \tilde{\wp} + \wp_0 = 0$ and $\wp' + \tilde{\wp}' = 0$ (with the evident notation $\wp' = \wp'(x_0)$, ...). Or else $\wp'^2 - \tilde{\wp}'^2 = 0$. Using the differential equation for $\wp$, $\wp'^2 = 4\wp^3 - g_2\wp - g_3$, one finds from here that

$$(\wp - \tilde{\wp})(4\wp^2 + 4\tilde{\wp}^2 + 4\wp\tilde{\wp} - g_2) = 0.$$

Since $\wp \neq \tilde{\wp}$ this implies that

$$0 = 4\wp^2 + 4\tilde{\wp}^2 + 4\wp\tilde{\wp} - g_2 = 4\wp^2 + 4\wp_0^2 + 4\wp\wp_0 - g_2$$

and

$$\wp(x_0+\omega') = -\tfrac{1}{2}\wp_0 \pm \tfrac{1}{2}\sqrt{g_2 - 3\wp_0^2}. \tag{B.6}$$

Hence, the positions x_1 and x_2 of the zeros of $f(x)$ are given by (5.16). In using these formulae it is necessary to keep in mind that $\wp$ is a multisheet function. Note that for $d = \omega$ one gets

$x_2 = 0$ and $x_1 = \omega$ since $\sqrt{g_2 - 3\wp_0^2} = e_2 - e_3$, $\wp^{-1}(e_3) = \omega'$, $\wp^{-1}(e_2) = \omega + \omega'$. Note also that since the equation $\wp(z) = A$ defines z up to a sign and up to the periods, either the sum or the difference of x_1 and x_2 modulo periods is equal to $-d$.

The function $p(x) = \pm 6\sqrt{f(x)}$ has simple zeros x_1 and x_2 and simple poles $1/p(x)$. It is not difficult to find the residues of $1/p(x)$ at these points

$$\operatorname*{Res}_{x=x_1} \frac{1}{p(x)} = -\operatorname*{Res}_{x=x_2} \frac{1}{p(x)} = \frac{1}{6\sqrt{g_2 - 3\wp_0^2}}. \tag{B.7}$$

This implies that the residues of $(a_1 - a_2)/p$ at the points $x_{1,2}$ are $\pm 1/2$.

Now, using the known [5] decomposition of an elliptic function in terms of ζ functions one finds

$$\frac{a_1 - a_2}{p} = \frac{1}{2}\zeta(x - x_1) - \frac{1}{2}\zeta(x - x_2) + b. \tag{B.8}$$

The constant b here may be calculated by the same formula at $x = \omega'$ where $1/p = 0$. This gives (5.17).

Using the fact that $\frac{\sigma'(z)}{\sigma(z)} = \zeta(z)$ one obtains

$$\frac{a_1 - a_2}{p} = \frac{1}{2}\left[\log\frac{\sigma(x - x_1)}{\sigma(x - x_2)}\right]' + b \tag{B.9}$$

and from (B.2) one finds

$$p = 6\left[\log\frac{\sigma(x + \omega' + d)}{\sigma(x + \omega')}\right]' - 6\zeta(d). \tag{B.10}$$

With the help of the last two relations one gets from (3.8)

$$[\log u_2^-(x)]' = \frac{1}{2}[\log p]' + \frac{1}{4}p + \frac{a_1 - a_2}{p}$$
$$= \frac{1}{2}[\log p]' + \frac{3}{2}\left[\log\frac{\sigma(x + \omega' + d)}{\sigma(x + \omega')}\right]' - \frac{3}{2}\zeta(d) + \frac{1}{2}\left[\log\frac{\sigma(x - x_1)}{\sigma(x - x_2)}\right]' + b. \tag{B.11}$$

It follows from here that (5.12) is a quadrature of the above formula. Similarly, formulae (5.13)–(5.15) follow from (5.12) and (3.9)–(3.11), and the proposition is proved.

References

[1] Dunne G and Feinberg J 1998 *Phys. Rev.* D **57** 1271
     Khare A and Sukhatme U 1999 *J. Math. Phys.* **40** 5473
[2] Fernández D J, Negro J and Nieto L M 2000 *Phys. Lett.* A **275** 338
[3] Fernández D J, Mielnik B, Rosas-Ortiz O and Samsonov B 2002 *Phys. Lett.* A **294** 168
[4] Fernández D J, Mielnik B, Rosas-Ortiz O and Samsonov B 2002 *J. Phys. A: Math. Gen.* **35** 4279
[5] Erdélyi A 1953 *Higher Trascendental Functions* (New York: McGraw-Hill)
[6] Gestezy F and Weikard R 1999 *Contemp. Math.* **221** 133
     Gestezy F and Holden H 2000 *J. Reine Angew. Math.* **527** 151
[7] Belokolos E D, Bobenko A I, Enolskii V Z and Its A R 1994 *Algebro-Geometric Approach to Nonlinear Integrable Equations* (Berlin: Springer)
[8] Matveev V B and Salle M A 1991 *Darboux Transformations and Solitons* (Berlin: Springer)
     Cooper F, Khare A and Sukhatme U 2001 *Supersymmetry in Quantum Mechanics* (Singapore: World Scientific)
[9] Sukumar C V 1985 *J. Phys. A: Math. Gen.* **18** 2917
     Sukumar C V 1985 *J. Phys. A: Math. Gen.* **18** 2937

Sukumar C V 1986 *J. Phys. A: Math. Gen.* **19** 2297
Sukumar C V 1987 *J. Phys. A: Math. Gen.* **20** 2461
Bagrov V G and Samsonov B F 1997 *Phys. Part. Nucl.* **28** 374
[10] Magnus W and Winkler S 1966 *Hill's Equation* (New York: Wiley)
[11] Trlifaj L 1989 *Inv. Probl.* **5** 1145
[12] Wolfram S 1996 *The Mathematica Book* (Cambridge: Wolfram Media)